Proving that polynomial kernel equation is simply a dot product of two vectors in a higher dimension.

Consider two 2D vectors $\vec{a}$ and $\vec{b}$ where,

$\vec{a} = [a_1, a_2]$, $\vec{b} = [b_1, b_2]$

Let the coefficient of polynomial,

$r = 1$

Let the degree of polynomial,

$d = 2$

Then kernel can be represented as

$$(a^T \cdot b + r)^d = (a^T \cdot b + 1)^2$$

$$= (a^T \cdot b + 1)(a^T \cdot b + 1)$$

$$= (a^T \cdot b)(a^T \cdot b) + 2\, a^T \cdot b + 1$$

$$= (a_1 b_1 + a_2 b_2)(a_1 b_1 + a_2 b_2) + 2(a_1 b_1 + a_2 b_2) + 1$$

since $a^T \cdot b = a_1 b_1 + a_2 b_2$

$$= a_1^2 b_1^2 + 2 a_1 a_2 b_1 b_2 + a_2^2 b_2^2 + 2 a_1 b_1 + 2 a_2 b_2 + 1 \quad \text{—— (1)}$$

Above result has 6 distinct terms, these can be disintegrated as dot product of two vectors representing $\vec{a}$ and $\vec{b}$ in a higher dimensional space. Lets do this one term at a time.

From (1)

$a_1^2 b_1^2 = a_1^2 \cdot b_1^2$

$2 a_1 b_1 a_2 b_2 = \sqrt{2}\, a_1 a_2 \cdot \sqrt{2}\, b_1 b_2$

$a_2^2 b_2^2 = a_2^2 \cdot b_2^2$

$2 a_1 b_1 = \sqrt{2}\, a_1 \cdot \sqrt{2}\, b_1$

$2 a_2 b_2 = \sqrt{2}\, a_2 \cdot \sqrt{2}\, b_2$

$1 = 1 \cdot 1$

We can rewrite (1) as

$$[a_1^2, \sqrt{2} a_1 a_2, a_2^2, \sqrt{2} a_1, \sqrt{2} a_2, 1] \cdot [b_1^2, \sqrt{2} b_1 b_2, b_2^2, \sqrt{2} b_1, \sqrt{2} b_2, 1]$$

The transformed vectors are:

$\vec{a} = [a_1^2, \sqrt{2}\, a_1 a_2, a_2^2, \sqrt{2}\, a_1, \sqrt{2}\, a_2, 1]$

$\vec{b} = [b_1^2, \sqrt{2}\, b_1 b_2, b_2^2, \sqrt{2}\, b_1, \sqrt{2}\, b_2, 1]$

$\vec{a}$ and $\vec{b}$ have been transformed from

$2D \rightarrow 6D$

Example

Lets consider

$\vec{a} = [1, 2]$ $\quad \vec{b} = [2, 1]$

with $r = 1$ and $d = 2$

these vectors when cast onto a higher dimension would look like

$\vec{a} = [1^2, \sqrt{2}(1)(2), 2^2, \sqrt{2}(1), \sqrt{2}(2), 1]$

$\vec{a} = [1, 2\sqrt{2}, 4, \sqrt{2}, 2\sqrt{2}, 1]$

$\vec{b} = [2^2, \sqrt{2}(2)(1), 1^2, \sqrt{2}(2), \sqrt{2}(1), 1]$

$\vec{b} = [4, 2\sqrt{2}, 1, 2\sqrt{2}, \sqrt{2}, 1]$

It is important to note that in reality only the dot product of the vectors in a higher dimension is calculated. These new higher dimension casted points are not stored.